Partial Fraction Decomposition (PFD)

General Concept: Consider a rational polynomial function $\frac{P(x)}{Q(x)}$ with $\deg(P(x)) < \deg(Q(x))$.

We can factor $Q(x)$ into

① linear polynomials in the form $ax+b$ with $a \neq 0$, and

② irreducible quadratic poly's in the form ax^2+b with $a>0$ and $b>0$.

A partial fraction decomposition (PFD) of $\frac{P(x)}{Q(x)}$ is a sum of terms that have a specific format that equals $\frac{P(x)}{Q(x)}$.

★ It can be proven that this PFD always exists!

How do we find the PFD of a rational polynomial function $\frac{P(x)}{Q(x)}$?

We need to factor the denominator $Q(x)$ completely into linear factors and irreducible quadratic factors.

Each factor of $Q(x)$ contributes terms to the PFD $f(x) = \frac{P(x)}{Q(x)}$ as follows:

⓪ Type of Factor in denominator $Q(x)$	: Term in PFD $f(x)$ contributed by factor.
① linear factor $(ax+b)$ in $Q(x)$	$f(x) = \frac{A}{ax+b} + (\cdots)$ for some constant A.
② repeated linear factor $(ax+b)^k$ in $Q(x)$ with k: positive integer	Each power of $(ax+b)$ contributes exactly 1 term. $f(x) = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k} + (\cdots)$ for some constants $A_1, A_2, \ldots, A_k$.
③ irreducible quadratic factor (ax^2+b) in $Q(x)$	$f(x) = \frac{Ax+B}{ax^2+b} + (\cdots)$ for some constants A, B.
④ repeated irreducible quadratic factor $(ax^2+b)^k$ in $Q(x)$	$f(x) = \frac{A_1x+B_1}{ax^2+b} + \frac{A_2x+B_2}{(ax^2+b)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+b)^k} + (\cdots)$ for some constants $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_k$

↑ linear factor ⇒ constant on numerator.

↑ irreducible quad. factor ⇒ linear term on numerator.

Example: The rational poly. function $\frac{x}{(x-2)(x+1)}$ has a PFD of $\frac{A}{x-2} + \frac{B}{x+1}$ with constants A and B.

How do we determine the constants of the PFD?

The PFD gives us the equality $\frac{P(x)}{Q(x)} = f(x)$.

We multiply both sides by the denominator $Q(x)$ and get an equality $\boxed{P(x) = f(x)\,Q(x)}$ between polynomials.

We have 2 methods in MTH 252, both using this equality.

Example: We have this PFD template $\frac{x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$;

Then, $\cancel{(x-2)(x+1)}\left[\frac{x}{\cancel{(x-2)(x+1)}}\right] = \left[\frac{A}{x-2} + \frac{B}{x+1}\right](x-2)(x+1)$

$$x = A(x+1) + B(x-2) \qquad (*)$$

Method 1. Elimination.

General Idea: The equality $P(x) = f(x)\,Q(x)$, when evaluated on any x-value, gives us another equality just in terms of the constants, i.e. without the x's.
So, we can choose <u>x-values strategically</u> so that we get equations in terms of 1 constant.

Example: $x = A(x+1) + B(x-2)$

If $x=-1$: $-1 = \cancel{A(-1+1)} + B(-1-2) = -3B$; $B = \frac{1}{3}$;

If $x=2$: $2 = A(2+1) + \cancel{B(2-2)} = 3A$; $A = \frac{2}{3}$;

Therefore, $\frac{x}{(x-2)(x+1)} = \left(\frac{2}{3}\right)\frac{1}{x-2} + \left(\frac{1}{3}\right)\frac{1}{x+1}$;

Method 2. Coefficient Matching.

General Idea: Two polynomials are equal if and only if the coefficient of each power of x agree.
So, we group $f(x)\,Q(x)$ in terms of powers of x and get a <u>linear system of equations</u> by equating the coefficients for each power of x.

Example. $x = A(x+1) + B(x-2) = Ax + A + Bx - 2B$

$= (1)x + (0) = (A+B)x + (A-2B) \Rightarrow$ linear system $\begin{cases} A+B = 1 & ① \\ A-2B = 0 & ② \end{cases}$

Use college algebra stuff to solve this system!

②: $A = 2B$; ①: $A + B = 2B + B = 3B = 1$, $B = \frac{1}{3}$; ②: $A = 2B = 2\left(\frac{1}{3}\right) = \frac{2}{3}$;

Therefore, $\frac{x}{(x-2)(x+1)} = \left(\frac{2}{3}\right)\frac{1}{x-2} + \left(\frac{1}{3}\right)\frac{1}{x+1}$;

Examples. Constants are determined using elimination.

① $\dfrac{x}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$;

$x = A(x+1) + B(x-2)$; If $x=-1$: $-1 = \cancel{A(-1+1)} + B(-1-2) = -3B$; $B = \frac{1}{3}$

If $x=2$: $2 = A(2+1) + \cancel{B(2-2)} = 3A$; $A = \frac{2}{3}$

Then, $\dfrac{x}{(x-2)(x+1)} = \left(\frac{2}{3}\right)\dfrac{1}{x-2} + \left(\frac{1}{3}\right)\dfrac{1}{x+1}$;

② $\dfrac{3x+2}{x^2-1} = \dfrac{3x+2}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$;

$3x+2 = A(x-1) + B(x+1)$; If $x=1$: $3(1)+2 = 5 = \cancel{A(1-1)} + B(1+1) = 2B$; $2B=5$; $B = \frac{5}{2}$;

If $x=-1$: $3(-1)+2 = -1 = A(-1-1) + \cancel{B(-1+1)} = -2A$; $-1=-2A$; $A = \frac{1}{2}$;

Then, $\dfrac{3x+2}{x^2-1} = \left(\frac{1}{2}\right)\dfrac{1}{x+1} + \left(\frac{5}{2}\right)\dfrac{1}{x-1}$;

③ $\dfrac{3x}{(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2}$;

$3x = A(x-2) + B$; If $x=2$: $3(2) = 6 = A(2-2) + B$; $B=6$;

If $x=0$: $3(0) = A(0-2) + 6 = -2A+6$; $0=-2A+6$; $2A=6$; $A=3$;

Then, $\dfrac{3x}{(x-2)^2} = \dfrac{3}{x-2} + \dfrac{6}{(x-2)^2}$;

④ $\dfrac{2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$;

$2 = A(x^2+1) + (Bx+C)(x+1)$;

If $x=-1$: $2 = A[(-1)^2+1] + \cancel{[B(-1)+C](-1+1)} = 2A$; $2A=2$; $A=1$;

If $x=0$: $2 = (1)(0^2+1) + (B(0)+C)(0+1) = 1+C$; $C+1=2$; $C=1$;

If $x=1$: $2 = (1)(1^2+1) + (B(1)+1)(1+1) = 2+2B+2$; $2B+4=2$; $2B=-2$; $B=-1$;

Then, $\dfrac{2}{(x+1)(x^2+1)} = \dfrac{1}{x+1} + \dfrac{-x+1}{x^2+1}$;

⑤ $\dfrac{2x+3}{(x+3)(x-1)(x+2)} = \dfrac{A}{x+3} + \dfrac{B}{x-1} + \dfrac{C}{x+2}$

$2x+3 = A(x-1)(x+2) + B(x+3)(x+2) + C(x+3)(x-1)$;

If $x=1$: $2(1)+3 = 5 = \cancel{A(0)(\cdots)} + B(1+3)(1+2) + \cancel{C(\cdots)(0)} = 12B$; $B = \frac{5}{12}$;

If $x=-2$: $2(-2)+3 = -4+3 = -1$

$= \cancel{A(\cdots)(0)} + \cancel{B(\cdots)(0)} + C(-2+3)(-2-1) = C(1)(-3) = -3C$; $C = \frac{-1}{-3} = \frac{1}{3}$;

If $x=-3$: $2(-3)+3 = -6+3 = -3$

$= A(-3-1)(-3+2) + \cancel{B(0)(\cdots)} + \cancel{C(0)(\cdots)} = A(-4)(-1) = 4A$; $A = \frac{-3}{4}$;

Then, $\dfrac{2x+3}{(x+3)(x-1)(x+2)} = \left(-\frac{3}{4}\right)\dfrac{1}{x+3} + \left(\frac{5}{12}\right)\dfrac{1}{x-1} + \left(\frac{1}{3}\right)\dfrac{1}{x+2}$;

How is this useful for integration? We can take antiderivatives of the terms of the PFD!

Some useful antiderivatives for your toolkit! You should be able to calculate these using previous concepts.

① $\int \frac{1}{ax+b}\,dx = \left(\frac{1}{a}\right)\ln|ax+b| + C$ using u-sub with $u=ax+b$

② If $k \geq 2$: $\int \frac{1}{(ax+b)^k}\,dx = \left(\frac{1}{a}\right)\left(\frac{1}{-k+1}\right)(ax+b)^{-k+1} + C$ using u-sub with $u=ax+b$

③ $\int \frac{x}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b| + C$ using u-sub with $u=ax^2+b$

④ $\int \frac{1}{ax^2+b}\,dx = \frac{1}{\sqrt{ab}}\arctan\left(x\sqrt{\frac{a}{b}}\right) + C$ using trig-sub with $\sqrt{a}\,x = \sqrt{b}\tan\theta$

Use ③ + ④ to split: $\int \frac{Ax+B}{ax^2+b}\,dx = \int A\left(\frac{x}{ax^2+b}\right) + B\left(\frac{1}{ax^2+b}\right)dx$

There are others!

Examples. Note: (1) PFDs of the integrand have been calculated on the prev. page.
(2) Some steps are not explicitly written here. Feel free to add some if you're confused.

① $\int \frac{x}{(x-2)(x+1)}\,dx = \int \left(\frac{2}{3}\right)\frac{1}{x-2} + \left(\frac{1}{3}\right)\frac{1}{x+1}\,dx = \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C$

② $\int \frac{3x+2}{x^2-1}\,dx = \int \left(\frac{1}{2}\right)\frac{1}{x+1} + \left(\frac{5}{2}\right)\frac{1}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{5}{2}\ln|x-1| + C$

③ $\int \frac{3x}{(x-2)^2}\,dx = \int \frac{3}{x-2} + \frac{6}{(x-2)^2}\,dx = 3\ln|x-2| - 6(x-2)^{-1} + C$

④ $\int \frac{2}{(x+1)(x^2+1)}\,dx = \int \frac{1}{x+1} + \frac{-x+1}{x^2+1}\,dx = \int \frac{1}{x+1} + (-1)\frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= \ln|x+1| - \frac{1}{2}\ln|x^2+1| + \arctan(x) + C$

⑤ $\int \frac{2x+3}{(x+3)(x-1)(x+2)}\,dx = \int \left(-\frac{3}{4}\right)\frac{1}{x+3} + \left(\frac{5}{12}\right)\frac{1}{x-1} + \left(\frac{1}{3}\right)\frac{1}{x+2}\,dx$

$= -\frac{3}{4}\ln|x+3| + \frac{5}{12}\ln|x-1| + \frac{1}{3}\ln|x+2| + C$

TIP: You can do the integration before identifying the constants.

Example: $I = \int \frac{x}{(x-2)(x+1)}\,dx = \int \frac{A}{x-2} + \frac{B}{x+1}\,dx = A\ln|x-2| + B\ln|x+1| + C$

After some calculation, we've found that $A = \frac{2}{3}$ and $B = \frac{1}{3}$;

Then, $I = \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C$